The Prime Number Theorem

Patrick Phillips

May 2020

Contents

1 Introduction

For the course of this paper let $\pi(x)$ denote the number of primes less than or equal to x. More explicitly we define $\pi(x) : \mathbb{R} \to \mathbb{Z}_{\geq 0}$ as

$$\pi(x) := \sum_{p \leq x} 1.$$

For example we can see that $\pi(10) = 4$, the cardinality of the set $\{2, 3, 5, 7\}$. Now the Prime Number Theorem states that $\frac{\pi(x)\ln(x)}{x} \to 1$ as $x \to \infty$. Or equivalently

$$\pi(x) \sim \frac{x}{\ln x}$$

where the notation $f(x) \sim g(x)$ means that $\lim_{x\to\infty} f(x)/g(x) = 1$.

The function $\pi(x)$ shown in Figure 1 can be thought of as a cumulative density function for the number of primes, and thus the Prime Number Theorem can be thought of as a statement about the distribution of primes for large N [1]. Since $\frac{d}{dx}\frac{x}{\ln(x)} \to \frac{1}{\ln(x)}$ as $x \to \infty$ we have that for large enough N, the probability that a random integer not greater than N is prime becomes very close to $1/\ln(N)$.

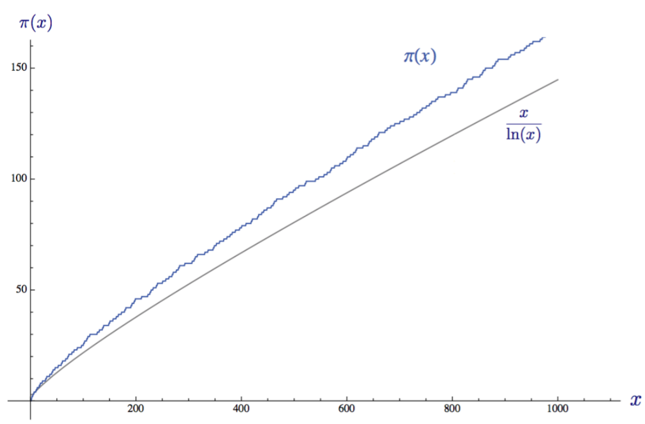

Figure 1: Plot of $\pi(x)$ and $x/\ln(x)$ [2].

We present the proof of the Prime Number Theorem by using the methods of complex analysis along with two important functions in the theory of primes, the Riemann zeta function and the Chebyshev function. In Section 2 we introduce the Riemann zeta function and prove some important results regarding its zeros and analyticity. Then in Section 3 we introduce the Chebyshev function and prove a reduction from the Prime Number Theorem to a statement in terms of the Chebyshev function. Finally in Section 4 we use this reduction and some interesting properties of these two functions to complete the proof of the Prime Number Theorem.

2 The Riemann Zeta Function

The Riemann zeta function is defined as

$$\zeta(z) := \sum_{n=1}^{\infty} \frac{1}{n^z}$$

with complex exponent n^z given by $n^z = e^{z \log n}$. For $z = 1$ this series is the harmonic series which diverges to $+\infty$. To see where this series converges notice that

$$|\zeta(z)| = \left| \sum_{n=1}^{\infty} \frac{1}{n^z} \right| \leq \sum_{n=1}^{\infty} \frac{1}{|n^z|} = \sum_{n=1}^{\infty} \frac{1}{n^x}$$

where $x = \text{Re(z)}$. By the integral test this last sum converges if the following integral converges:

$$\int_{t=1}^{\infty} \frac{1}{t^x} dt = \left[\frac{1}{1-x} t^{1-x} \right]_{x=1}^{\infty}$$

which we can see will be on $x = \text{Re(z)} > 1$.

2.1 Euler's Product Formula

Theorem 1 For Re(z) > 1 the Riemann zeta function is given by the product

$$\prod_{j=1}^{\infty} \left(\frac{1}{1 - p_j^{-z}} \right)$$

where $\{p_j\}$ is the sequence of increasing prime numbers $\{2, 3, 5, ...\}$. Furthermore ζ is zero free and analytic on Re(z) > 1.

Proof: First notice that we can write

$$\begin{aligned}
(1 - p_j^{-z})(1 + p_j^{-z} + p_j^{-2z} + ... + p_j^{-(nz-1)} + p_j^{-nz}) \\
&= 1 + p_j^{-z} + p_j^{-2z} + ... + p_j^{-(nz-1)} + p_j^{-nz} \\
&\quad - p_j^{-z} - p_j^{-2z} - ... - p_j^{-(nz-1)} - p_j^{-nz} - p_j^{-(nz+1)} \\
&= 1 - p_j^{-(nz+1)}
\end{aligned}$$

which rearranging terms yields

$$\frac{1}{1-p_j^{-z}} - (1 + p_j^{-z} + p_j^{-2z} + ... + p_j^{-(nz-1)} + p_j^{-nz}) = \frac{p_j^{-(nz+1)}}{1-p_j^{-z}}.$$

If Re(z) > 1 the right hand side in the equation above goes to 0 as $n \to \infty$ leaving us with

$$\begin{aligned}\frac{1}{1-p_j^{-z}} &= 1 + p_j^{-z} + p_j^{-2z} + ... \\ &= 1 + \frac{1}{p_j^z} + \frac{1}{p_j^{2z}} + ... \qquad .\end{aligned}$$

Now consider the partial product

$$\prod_{j=1}^{m} \frac{1}{1-p_j^{-z}} = \prod_{j=1}^{m}(1 + \frac{1}{p_j^z} + \frac{1}{p_j^{2z}} + ...).$$

If we actually multiply out the finitely many *converging* series on the right hand side, we will be left with a sum of terms in the form of $\frac{1}{1^i p_1^j p_2^k ... p_m^l}$ with all infinitely many possible combinations of $\{i, j, k, ...l\}$. But by the Fundamental Theorem of Algebra, we know each integer has a unique factorization over the primes, so each term's denominator is just some *unique* positive integer whose factorization is over the first m primes. We can write this as

$$\prod_{j=1}^{m}(1 + \frac{1}{p_j^{-z}} + \frac{1}{p_j^{-2z}} + ...) = \sum_{i,j,k,...l} \frac{1}{1^i p_1^j p_2^k ... p_m^l} = \sum_{n \in P_m} \frac{1}{n^z}$$

where P_m consists of 1 along with the positive integers whose factorization is in the set $\{p_1, ...p_m\}$. Thus letting $m \to \infty$ we have that for Re(z) > 1

$$\prod_{j=1}^{\infty} \frac{1}{1-p_j^{-z}} = \sum_{n=1}^{\infty} \frac{1}{n^z}.$$

Because both the above series and product converge on Re(z) > 1, the product representation proves that ζ is analytic and nonzero on Re(z) > 1 since for all j we have that $(1 - p_j^{-z})^{-1}$ is analytic and nonzero and an infinite product of analytic and nonzero functions will be analytic and nonzero as long as it converges. ■

2.2 Extending the Zeta Function

In order to make use of ζ in later sections we will need an analytic extension to the right half plane Re(z) > 0.

Theorem 2 ζ has an extension to Re(z) > 0 and $z \neq 1$ with a simple pole at $z = 1$ with residue 1.

Proof: First recall the summation by parts formula which states that for sequences $\{a_n\}$ and $\{b_n\}$ we have

$$\sum_{n=r}^{s} a_n \Delta b_n = a_{s+1}b_{s+1} - a_r b_r - \sum_{n=r}^{s} b_{n+1}\Delta a_n \tag{1}$$

where $\Delta a_n = a_{n+1} - a_n$. Using sequences $a_n = n$ and $b_n = \frac{1}{n^z}$ the summation by parts formula yields

$$\sum_{n=1}^{k-1} n\left[\frac{1}{(n+1)^z} - \frac{1}{n^z}\right] = \frac{1}{k^{z-1}} - 1 - \sum_{n=1}^{k-1}\frac{1}{(n+1)^z}.$$

Rearranging gives

$$1 + \sum_{n=1}^{k-1}\frac{1}{(n+1)^z} = \frac{1}{k^{z-1}} - \sum_{n=1}^{k-1} n\left[\frac{1}{(n+1)^z} - \frac{1}{n^z}\right]. \tag{2}$$

Now note that we also can write this term as an integral

$$n\left[\frac{1}{(n+1)^z} - \frac{1}{n^z}\right] = -nz\int_n^{n+1} t^{-z-1}dt = -z\int_n^{n+1}[t]t^{-z-1}dt$$

where $[t]$ denotes the largest integer less than or equal to t. Plugging this into Eq. 2 we can write

$$\begin{aligned}\sum_{n=1}^{k}\frac{1}{n^z} = 1 + \sum_{n=1}^{k-1}\frac{1}{(n+1)^z} &= \frac{1}{k^{z-1}} + z\sum_{n=1}^{k-1}\int_n^{n+1}[t]t^{-z-1}dt \\ &= \frac{1}{k^{z-1}} + z\int_1^k [t]t^{-z-1}dt.\end{aligned}$$

As $k \to \infty$, we have the integral

$$\zeta(z) = z\int_1^\infty [t]t^{-z-1}dt$$

for $\mathrm{Re}(z) > 1$. Now consider the similar integral

$$z\int_1^\infty tt^{-z-1}dt = z\int_1^\infty t^{-z}dt = \frac{z}{z-1} = 1 + \frac{1}{z-1}.$$

Combining the last two equations we can write

$$\zeta(z) = 1 + \frac{1}{z-1} + z\int_1^\infty ([t]-t)t^{-z-1}dt. \tag{3}$$

Now fix $k > 1$ and consider the integral $\int_1^k ([t]-t)t^{-z-1}dt$. By Theorem 9 in the Appendix, this integral is an entire function of z. Furthermore, if $\mathrm{Re}(z) > 0$, then

$$|\int_1^k ([t]-t)t^{-z-1}dt| \leq \int_1^k t^{-\mathrm{Re}(z+1)}dt \leq \int_1^\infty t^{-1-\mathrm{Re}(z)}dt = \frac{1}{\mathrm{Re}(z)}.$$

This implies that the sequence $f_k(z) = \int_1^k ([t]-t)t^{-z-1}dt$ of analytic functions on $\mathrm{Re}(z) > 0$ is uniformly bounded on compact subsets. Hence by Vitali's Theorem (11 in the Appendix), the limit function

$$f(z) = \int_1^\infty ([t]-t)t^{-z-1}dt$$

(as the uniform limit on compact subsets of $\mathrm{Re}(z) > 0$) is analytic, and using Eq. 3 we have that

$$\zeta(z) = 1 + \frac{1}{z-1} + z\int_1^\infty ([t]-t)t^{-z-1}dt \tag{4}$$

is analytic on $\{z : \mathrm{Re(z)} > 0, \mathrm{z} \neq 1\}$ and has a simple pole with residue 1 at $z = 1$. ■

2.3 Zeros of the Zeta Function

The Euler product representation of ζ given in Theorem 1 lets us conclude that ζ has no zeros in $\mathrm{Re}(z) > 1$. But what about the zeros of ζ in $\mathrm{Re}(z) \leq 1$? We examine this question in more detail in Section 6, but here we will first prove that ζ has no zeros on the line $\mathrm{Re}(z) = 1$.

Theorem 3 The Riemann zeta function has no zeros on $\mathrm{Re}(z) = 1$, so $(z-1)\zeta(z)$ is analytic and zero-free on a neighborhood of $\mathrm{Re}(z) \geq 1$.

Proof: Fix a real number $y \neq 0$ and consider the auxiliary function

$$h(x) = \zeta^3(x)\zeta^4(x+iy)\zeta(x+i2y)$$

where $x \in \mathbb{R}$ and $x > 1$. Now consider $\ln|\zeta(z)|$ for $\mathrm{Re}(z) > 1$. By Euler's product formula we have

$$\begin{aligned}\ln|\zeta(z)| &= -\sum_{j=1}^\infty \ln|1-p_j^{-z}| \\ &= -\mathrm{Re}\sum_{j=1}^\infty \mathrm{Log}(1-p_j^{-z}) \\ &= \mathrm{Re}\sum_{j=1}^\infty\sum_{n=1}^\infty \frac{1}{n}p_j^{-nz}\end{aligned}$$

where in the second line we use the definition of Log(z) = ln|z| + iArg(z) to rewrite in terms of the real part of the principal logarthim, and then in the last line we use the expansion $-\text{Log}(1-w) = \sum_{n=1}^{\infty} w^n/n$, valid for $|w| < 1$. Applying similar analysis to $\ln|h(x)|$ we have

$$\begin{aligned}
\ln|h(x)| =& 3\ln|\zeta(x)| + 4\ln|\zeta(x+iy)| + \ln|\zeta(x+i2y)| \\
=& 3\text{Re}\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{n}p_j^{-nx} + 4\text{Re}\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{n}p_j^{-nx-iny} \\
&+ \text{Re}\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{n}p_j^{-nx-i2ny} \\
=& \sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{n}p_j^{-nx}\text{Re}(3 + 4p_j^{-iny} + p_j^{-i2ny}).
\end{aligned}$$

Now note that $p_j^{-iny} = e^{-iny\ln p_j}$ and $p_j^{-i2ny} = e^{-i2ny\ln p_j}$. Thus taking $\theta = -ny\ln(p_j)$ we can see that $\text{Re}(3 + 4p_j^{-iny} + p_j^{-i2ny})$ has the form

$$3 + 4\cos\theta + \cos 2\theta = 3 + 4\cos\theta + 2\cos^2\theta - 1 = 2(1+\cos\theta)^2 \geq 0.$$

Since $p_j^{-nx}/n \geq 0$ as well, we have that $\ln|h(x)| \geq 0$. Thus

$$|h(x)| = |\zeta^3(x)||\zeta^4(x+iy)||\zeta(x+i2y)| \geq 1.$$

So we have

$$\frac{|h(x)|}{x-1} = |(x-1)\zeta(x)|^3|\frac{\zeta(x+iy)}{x-1}|^4|\zeta(x+i2y)| \geq \frac{1}{x-1}.$$

Now suppose $\zeta(1+iy) = 0$. Considering the limit as $x \to 1^+$, we have that the the left hand side of this inequality would approach a finite limit $|\zeta'(1+iy)|^4|\zeta(1+i2y)|$ as $x \to 1^+$ since ζ has a simple pole at 1 with residue 1. However, the right hand side of the inequality contradicts this. We conclude that $\zeta(1+iy) \neq 0$. Since y is an arbitrary nonzero real number, ζ cannot have any zeros on $\text{Re}(z) = 1$. ■

3 The Chebyshev Function

We will now introduce two new functions, the Von Mangoldt and Chebyshev functions, that are explicitly defined in terms of prime numbers. In Section 3.1 we will in fact show an equivalency between the Prime Number Theorem and the asymptotic behavior of the Chebyshev function. In Section 3.2 we will find a necessary upper bound on the asymptotic behavior of the Chebyshev function. Then in section 3.3 we will find a connection between the zeta function and

Chebyshev function that will turn out to be essential to our proof of the Prime Number Theorem.

First define the Von Mangoldt function

$$\Lambda(n) = \begin{cases} \ln p & \text{if } n = p^m \text{ for some } m, \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\Lambda(n)$ is $\ln p$ if n is a power of the prime p, and is 0 if not. Next define the Chebyshev function ψ on $x \geq 0$ as

$$\psi(x) = \sum_{n \leq x} \Lambda(n).$$

Notice that we can solve for the Von Mangoldt function as

$$\Lambda(k) = \psi(k) - \psi(k-1). \tag{5}$$

An equivalent expression for ψ is

$$\psi(x) = \sum_{p \leq x} m_p(x) \ln p \tag{6}$$

where the sum is over primes $p \leq x$ and $m_p(x)$ is the largest integer such that $p^{m_p(x)} \leq x$. For example, $\psi(10.4) = 3\ln 2 + 2\ln 3 + \ln 5 + \ln 7$. Note that $p^{m_p(x)} \leq x$ iff $m_p(x) \ln p \leq \ln x$ iff $m_p(x) \leq \dfrac{\ln x}{\ln p}$. Thus $m_p(x) = \left[\frac{\ln x}{\ln p}\right]$ where as before, [ ] denotes the greatest integer function.

3.1 Chebyshev's Reduction of the Prime Number Theorem

We will now find a statement equivalent to the Prime Number Theorem in terms the Chebyshev function $\psi(x)$ that we have defined.

Theorem 4 The Prime Number Theorem holds, that is $\pi(x) \ln x / x \to 1$, iff $\psi(x)/x \to 1$ as $x \to \infty$.

Proof: Using our definition of $\psi(x)$ in Eq. 6 we have

$$\begin{aligned} \psi(x) &= \sum_{p \leq x} [\frac{\ln x}{\ln p}] \ln p \\ &\leq \sum_{p \leq x} \frac{\ln x}{\ln p} \ln p \\ &= \ln x \sum_{p \leq x} 1 \\ &= (\ln x)\pi(x). \end{aligned} \tag{7}$$

And for $1 < y < x$, we have

$$\begin{aligned}
\pi(x) &= \pi(y) + \sum_{y<p\leq x} 1 \\
&\leq \pi(y) + \sum_{y<\text{p}\leq x} \frac{\ln p}{\ln y} \\
&< y + \frac{1}{\ln y} \sum_{y<p\leq x} \ln p \\
&\leq y + \frac{1}{\ln y}\psi(x). \qquad (8)
\end{aligned}$$

Now taking $y = x/(\ln x)^2$ in Eq. 8, we get

$$\pi(x) \leq \frac{x}{(\ln x)^2} + \frac{1}{\ln x - 2\ln\ln x}\psi(x).$$

Multiplying each side by $\ln(x)/x$,

$$\pi(x)\frac{\ln x}{x} \leq \frac{1}{\ln x} + \frac{\ln x}{\ln x - 2\ln\ln x}\frac{\psi(x)}{x}.$$

Now from Eq. 7 and above we have,

$$\frac{\psi(x)}{x} \leq \frac{\ln x}{x}\pi(x) \leq \frac{1}{\ln x} + \frac{\ln x}{\ln x - 2\ln\ln x}\frac{\psi(x)}{x}.$$

Consider the limit as $x \to \infty$. On the rightmost side $1/\ln(x) \to 0$ and $\ln(x)/[\ln(x) - 2\ln\ln(x)] \to 1$. Thus we see that $\psi(x)/x \to 1$ iff $\pi(x)\ln x/x \to 1$ as $x \to \infty$. ■

Of course the goal will now be to show that $\psi(x)/x \to 1$ as $x \to \infty$. A necessary intermediate step for our proof will be to establish the following bound on the asymptotic behavior of $\psi(x)$.

3.2 Upper Bound on Asymptotic Behavior of $\psi(x)$

Theorem 5 $\psi(x) = O(x)$, that is there exists $C > 0$ and $x > x_0$ such that $\psi(x) \leq Cx$, for all $x > x_0$.

Proof: Again recall our definition $\psi(x) = \sum_{p\leq x} \left[\frac{\ln x}{\ln p}\right] \ln p, x > 0$. Fix $x > 0$ and let m be an integer such that $2^m < x \leq 2^{m+1}$ Then we have that

$$\begin{aligned}
\psi(x) &= \psi(2^m) + \psi(x) - \psi(2^m) \\
&\leq \psi(2^m) + \psi(2^{m+1}) - \psi(2^m) \\
&= \sum_{p\leq 2^m} \left[\frac{\ln 2^m}{\ln p}\right] \ln p + \sum_{2^m<p\leq 2^{m+1}} \left[\frac{\ln 2^{m+1}}{\ln p}\right] \ln p. \qquad (9)
\end{aligned}$$

Consider for any positive integer n,

$$\sum_{n<p\le 2n} \ln p = \ln \prod_{n<p\le 2n} p \quad .$$

Now fix an arbitrary prime p such that $n < p \le 2n$. Clearly p divides $(2n)!/n! = n!\binom{2n}{n}$. Since such a p cannot divide $n!$ (all of $n!$'s divisors are less than p), it follows that p divides $\binom{2n}{n}$. Thus by unique factorization and the fact that p divides $\binom{2n}{n}$ we have

$$\begin{aligned} \prod_{n<p\le 2n} p &\le \binom{2n}{n} \\ &< (1+1)^{2n} && \text{(by the Binomial Theorem)} \\ &= 2^{2n} \end{aligned}$$

and we arrive at

$$\sum_{n<p\le 2n} \ln p < \ln(2^{2n}) = 2n \ln 2. \tag{10}$$

Thus from Eq. 10 we have

$$\sum_{p\le 2^m} \ln p = \sum_{k=1}^{\mathfrak{m}} \Big(\sum_{2^{k-1}<p\le 2^k} \ln p \Big) < \sum_{k=1}^{m} 2^k \ln 2 < 2^{m+1} \ln 2$$

as well as

$$\sum_{2^m<p\le 2^{m+1}} \ln p < 2^{m+1} \ln 2.$$

Suppose $p \le x$ is such that $\left[\frac{\ln x}{\ln p}\right] > 1$, then $\frac{\ln x}{\ln p} \ge 2$. Thus we would have $x \ge p^2$ and $\sqrt{x} \ge p$. This means the terms in the sum $\sum_{p\le x} \left[\frac{\ln x}{\ln p}\right] \ln p$ where $\left[\frac{\ln x}{\ln p}\right] > 1$ occur only when $p \le \sqrt{x}$, and the sum of terms of this form can contribute no more than

$$\sum_{p\le \sqrt{x}} \frac{\ln x}{\ln p} \ln p = \pi(\sqrt{x}) \ln x.$$

Thus if $2^m < x \le 2^{m+1}$

$$\begin{aligned}
\psi(x) &\le \sum_{p \le 2^m} \left[\frac{\ln 2^m}{\ln p}\right] \ln p + \sum_{2^m < p \le 2^{m+1}} \left[\frac{\ln 2^{m+1}}{\ln p}\right] \ln p && \text{(by Eq. 9)} \\
&\le 2^{m+1} \ln 2 + 2^{m+1} \ln 2 + \pi(\sqrt{x}) \ln x && \text{(by Eq. 10)} \\
&= 2^{m+2} \ln 2 + \pi(\sqrt{x}) \\
&< 4x \ln 2 + \pi(\sqrt{x}) \ln x && \text{(since by assumption } x > 2^m) \\
&\le 4x \ln 2 + \sqrt{x} \ln x \\
&= (4 \ln 2 + \frac{1}{\sqrt{x}} \ln x) x.
\end{aligned}$$

Since $\frac{1}{\sqrt{x}} \ln x \to 0$ as $x \to \infty$, we have $\psi(x) = O(x)$. ■

3.3 Relating the Zeta and Chebyshev Functions

We now return to our hero, the Riemann zeta function, or more specifically, we will look at the negative logarithmic derivative $-\zeta'/\zeta$. By Theorem 3 we know ζ is analytic on a neighborhood of $\{z : \text{Re}z \ge 1 \text{ and } z \ne 1\}$, and by standard theorems in complex analysis, this implies all derivatives of ζ are also analytic in this domain. Thus with the additional fact that ζ has no zeros on a neighborhood of $\{z : \text{Re}z \ge 1 \text{ and } z \ne 1\}$ (also by Theorem 3), the function $-\zeta'/\zeta$ is analytic on this domain.

Theorem 6 For Re(z) > 1, $-\zeta'/\zeta$ is the *Mellin Transform* of ψ. That is

$$-\frac{\zeta'(z)}{\zeta(z)} = z \int_1^\infty \psi(t) t^{-z-1} dt.$$

Proof: If $\text{Re}(z) > 1$, by Theorem 1 we have $\zeta(z) = \prod_p (1 - p^{-z})^{-1}$. Applying the product rule between the first term with prime p and the rest of the terms we get,

$$\begin{aligned}
\zeta'(z) &= \frac{d}{dz}\left[\frac{1}{1-p^{-z}}\right] * \prod_{q \ne p} \frac{1}{1-q^{-z}} + \frac{1}{1-p^{-z}} * \frac{d}{dz}\left[\prod_{q \ne p} \frac{1}{1-q^{-z}}\right] \\
&= \frac{-p^{-z} \ln p}{(1-p^{-z})^2} * \prod_{q \ne p} \frac{1}{1-q^{-z}} + \frac{1}{1-p^{-z}} * \frac{d}{dz}\left[\prod_{q \ne p} \frac{1}{1-q^{-z}}\right].
\end{aligned}$$

And we can apply the same product rule decomposition on the second term, noting that the factor of $(1 - p^{-z})$ can just be added to the resulting product and obtain the expression

$$\begin{aligned}\zeta'(z) &= \sum_p \frac{-p^{-z}\ln p}{(1-p^{-z})^2}\prod_{q\neq p}\frac{1}{1-q^{-z}}\\ &= \zeta(z)\sum_p \frac{-p^{-z}\ln p}{(1-p^{-z})^2}(1-p^{-z})\\ &= \zeta(z)\sum_p \frac{-p^{-z}\ln p}{1-p^{-z}}.\end{aligned}$$

Thus we have the expression for ζ'/ζ,

$$-\frac{\zeta'(z)}{\zeta(z)} = \sum_p \frac{p^{-z}\ln p}{1-p^{-z}} = \sum_p\sum_{n=1}^{\infty} p^{-nz}\ln p.$$

Using similar analysis to that of Section 2, we can see that the iterated sum is absolutely convergent for $\mathrm{Re}(z) > 1$. Thus we can rearrange this as the double sum

$$\begin{aligned}-\frac{\zeta'(z)}{\zeta(z)} &= \sum_{(p,n),n\geq 1}(p^n)^{-z}\ln p\\ &= \sum_k k^{-z}\ln p, \qquad \text{where } k = p^n \text{ for some } n\\ &= \sum_{k=1}^{\infty} k^{-z}\Lambda(k) \qquad \text{(definition of the Von Mangoldt function!)}\\ &= \sum_{k=1}^{\infty} k^{-z}(\psi(k)-\psi(k-1)). \qquad \text{(by Eq. 5)}\end{aligned}$$

Now by using summation by parts again from Eq. 1, this time with $a_k = k^{-z}$, $b_{k+1} = \psi(k)$, and $b_1 = \psi(0) = 0$, we can write

$$\sum_{k=1}^{M} k^{-z}(\psi(k)-\psi(k-1)) = \psi(M)(M+1)^{-z} + \sum_{k=1}^{M}\psi(k)(k^{-z}-(k+1)^{-z}). \tag{11}$$

Considering just the last term above we have

$$\begin{aligned}\sum_{k=1}^{M}\psi(k)(k^{-z}-(k+1)^{-z}) &= \sum_{k=1}^{M}\psi(k)z\int_{k}^{k+1}t^{-z-1}dt \\ &= \sum_{k=1}^{M}z\int_{k}^{k+1}\psi(t)t^{-z-1}dt \\ &= z\int_{1}^{M}\psi(t)t^{-z-1}dt\end{aligned}$$

where in the second line we use that ψ is constant on each interval $[k,\ k+1)$.

Now consider the limit as $M \to \infty$ of Eq. 11. The left hand side just becomes $-\zeta'/\zeta$ by the derivation on the previous page. From the definition of ψ in Eq. 6 we have $\psi(x) \leq x \ln x$, so if $\mathrm{Re}(z) > 1$ we have $\psi(M)(M+1)^{-z} \to 0$. Finally the last term we have just found our expression for. Thus we have for Re(z) > 1,

$$-\frac{\zeta'(z)}{\zeta(z)} = z\int_{1}^{\infty}\psi(t)t^{-z-1}dt. \qquad \blacksquare$$

4 Proof of the Prime Number Theorem

Our plan is to prove a "Tauberian" Theorem in Section 4.1, and then a vital corollary regarding the Mellin Transform in Section 4.2. With this corollary we will deduce the Prime Number Theorem.

4.1 Tauberian Theorem

Theorem 7 Let F be bounded and piecewise continuous on $[0,\ +\infty)$. Then its Laplace transform

$$G(z) = \int_{0}^{\infty}F(t)e^{-zt}dt$$

exists and is analytic on $\mathrm{Re}(z) > 0$. Furthermore, assume that G has an analytic extension to a neighborhood of the imaginary axis, $\mathrm{Re}(z) = 0$. Then $\int_0^\infty F(t)dt$ exists as an improper integral and is equal to $G(0)$ (In fact, $\int_0^\infty F(t)e^{-iyt}dt$ converges for every $y \in \mathbb{R}$ to $G(iy)$).

Proof: Suppose F is bounded and piecewise continuous on $[0,\ +\infty)$. Then for $0 < \lambda < \infty$ define,

$$G_\lambda(z) = \int_{0}^{\lambda}F(t)e^{-zt}dt.$$

By Theorem 9 in the Appendix, each function G_λ is entire. Now consider the modulus of each G_λ.

$$
\begin{aligned}
|G(z)| &= \left| \int_0^\lambda F(t)e^{-zt}dt \right| \\
&\le \int_0^\lambda |F(t)e^{-zt}|dt && \text{(see Appendix Theorem 10)} \\
&= \int_0^\lambda |F(t)||e^{-xt}||e^{-iyt}|dt \\
& && \text{(splitting the modulus and taking } z = x+iy) \\
&\le \int_0^\infty |F(t)|e^{-xt}dt && (e^{-xt} \text{ is real and } |e^{-iyt}| = 1) \\
&\le \int_0^\infty e^{-xt}dt && \text{(by assumption } |F(t)| = 1) \\
&= \frac{e^{-\lambda x}}{x} \\
&\le \frac{1}{x} = \frac{1}{\text{Re}(z)}.
\end{aligned}
$$

This shows each G_λ is bounded on Re(z) > 0. Thus by Vitali's Theorem (11 in the Appendix), F's Laplace transform G is defined and analytic on $\text{Re}(z) > 0$. Assume, per hypothesis of the theorem, that G has been extended to an analytic function on a region containing $\text{Re}(z) \ge 0$. Since F is bounded we will also assume (only for convenience) that $|F(t)| \le 1, t \ge 0$.

Notice that the conclusion of our theorem can be expressed as

$$
\lim_{\lambda \to \infty} G_\lambda(0) = G(0).
$$

That is, the improper integral $\int_0^\infty F(t)dt$ exists and converges to $G(0)$. We will show that $|G_\lambda(0) - G(0)|$ as $\lambda \to \infty$ proving our theorem. To do so, we first use Cauchy's integral formula around a closed contour centered at 0 to get an estimate of $|G_\lambda(0) - G(0)|$. For each $R > 0$, let $\delta(R) > 0$ be so small that G is analytic inside and on the closed path γ_R shown in Figure 2. (Note that since G is analytic on an open set containing $\text{Re}(z) \ge 0$, such a $\delta(R) > 0$ must exist.) We have by Cauchy's integral formula,

$$
G(0) - G_\lambda(0) = \frac{1}{2\pi i} \int_{\gamma_R} (G(z) - G_\lambda(z)) \frac{1}{z} dz. \tag{12}
$$

We will have to split this contour integral into two pieces, paying particular attention to the integration along the path in $\text{Re}(z) < 0$ since we only have G explicitly defined on the right half plane. Let γ_R^+ denote that portion of γ_R that lies in $\text{Re}(z) > 0$, and γ_R^- the portion that lies in $\text{Re}(z) < 0$.

We will first bound the integral along γ_R^+. As a first stab, let's try applying the ML inequality (12 in the Appendix) to bound the integral on the right hand

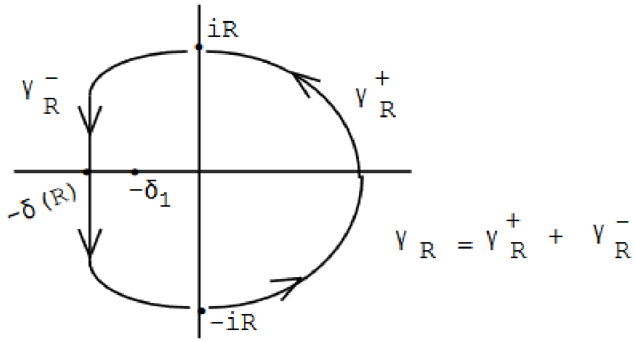

Figure 2: Contour integral path of γ_R [3].

side of the equation above. If we can show that this bound goes to 0 as $\lambda \to \infty$ our work for this half of the contour will be done. For $z \in \gamma_R^+$ and $x = \text{Re}(z)$, we have

$$\begin{aligned}
\left|\frac{G(z) - G_\lambda(z)}{z}\right| &= \frac{1}{|z|}\left|\int_0^\infty F(t)e^{-zt}dt - \int_0^\lambda F(t)e^{-zt}dt\right| \\
&= \frac{1}{R}\left|\int_\lambda^\infty F(t)e^{-zt}dt\right| && (|z| = R \text{ for } z \in \gamma_R^+) \\
&\leq \frac{1}{R}\int_\lambda^\infty |F(t)e^{-zt}|dt && (\text{see Appendix Theorem 10}) \\
&\leq \frac{1}{R}\int_\lambda^\infty e^{-xt}dt && (\text{by assumption } |F(t)| = 1) \\
&= \frac{1}{R}\frac{e^{-\lambda x}}{x} && (13) \\
&\leq \frac{1}{R}\frac{1}{x} = \frac{1}{R}\frac{1}{\text{Re}(z)}.
\end{aligned}$$

Unfortunately, $1/\text{Re}(z)$ is unbounded on γ_R^+, as the contour approaches Re(z) = 0. However, with some clever modifications of the contour integral in Eq. 12, we can still find our desired bound.

$$\begin{aligned}
&G(0) - G_\lambda(0) = [G(0) - G_\lambda(0)]e^{\lambda z} \\
&= \frac{1}{2\pi i}\int_{\gamma_R}(G(z) - G_\lambda(z))e^{\lambda z}\frac{1}{z}dz && (\text{Cauchy's Integral Formula}) \\
&= \frac{1}{2\pi i}\int_{\gamma_R}(G(z) - G_\lambda(z))e^{\lambda z}\frac{1}{z}dz + \frac{1}{2\pi i}\int_{\gamma_R}(G(z) - G_\lambda(z))e^{\lambda z}\frac{z}{R^2}dz \\
&= \frac{1}{2\pi i}\int_{\gamma_R}(G(z) - G_\lambda(z))e^{\lambda z}\left(\frac{1}{z} + \frac{z}{R^2}\right)dz.
\end{aligned}$$

It's clear that we are still justified to use Cauchy's Integral formula in the second line since $e^{\lambda z}$ is entire. In the third line we use the fact that each of the functions in the second integrand is analytic inside γ_R, thus by Cauchy's Theorem this second integral is 0, and we are free to add it on.

Now note that for $|z| = R$ we have

$$\frac{1}{z} + \frac{z}{R^2} = \frac{\overline{z}}{|z|^2} + \frac{z}{R^2} = \frac{2\text{Re}(z)}{R^2}.$$

so if $z \in \gamma_R^+$ we have by the previous line and Eq. 13

$$|(G(z) - G_\lambda(z))e^{\lambda z}\left(\frac{1}{z} + \frac{z}{R^2}\right)| \le \frac{1}{\text{Re}(z)}e^{-\lambda\text{Re}(z)}e^{\lambda\text{Re}(z)}\frac{2\text{Re}(z)}{R^2} = \frac{2}{R^2}.$$

Thus the ML inequality yields

$$\left|\frac{1}{2\pi i}\int_{\gamma_R^+}(G(z) - G_\lambda(z))e^{\lambda z}\left(\frac{1}{z} + \frac{z}{R^2}\right)dz\right| \le \frac{1}{R}.$$

Note that this estimate of the integral along the path γ_R^+ is independent of λ, however we are free to choose our contour and thus R to make this portion of the integral vanish.

Now let us consider the contribution to the integral along γ_R of the integral along γ_R. First we use the triangle inequality to obtain the estimate

$$\begin{aligned}&\left|\frac{1}{2\pi i}\int_{\gamma_R^-}(G(z) - G_\lambda(z))e^{\lambda z}\left(\frac{1}{z} + \frac{z}{R^2}\right)dz\right| \\ &\qquad \le \left|\frac{1}{2\pi i}\int_{\gamma_R^-}G(z)e^{\lambda z}\left(\frac{1}{z} + \frac{z}{R^2}\right)dz\right| + \left|\frac{1}{2\pi x}\int_{\gamma_R^-}G_\lambda(z)e^{\lambda z}\left(\frac{1}{z} + \frac{z}{R^2}\right)dz\right| \\ &\qquad = |I_1(R)| + |I_2(R)|.\end{aligned}$$

First consider $I_2(R)$. Since $G_\lambda(z)$ in this case is an entire function (only G is not), we can replace the path of integration γ_R^- by the semicircular path from iR to $-iR$ in the left half plane. For z on this semicircular arc, the modulus of the integrand in $I_2(R)$ is

$$\begin{aligned}\left|G_\lambda(z)e^{\lambda z}\left(\frac{1}{z}+\frac{z}{R^2}\right)\right| &= \left|(\int_0^\lambda F(t)e^{-zt}dt)e^{\lambda z}\frac{2\mathrm{Re}(z)}{R^2}\right| \\ &\leq \left|\int_0^\lambda F(t)e^{-zt}dt\right| |e^{\lambda z}||\frac{2\mathrm{Re}(z)}{R^2}| \\ &\leq \int_0^\lambda |F(t)e^{-zt}|dt\, e^{\lambda x}\frac{2|\mathrm{Re}(z)|}{R^2} && \text{(by Theorem 10)} \\ &\leq \int_0^\infty e^{-xt}dt\, e^{\lambda x}\frac{2|\mathrm{Re}(z)|}{R^2} && \text{(assumption } |F(t)|\leq 1) \\ &\leq \frac{1}{|\mathrm{Re}(z)|}e^{\lambda x}\frac{2|\mathrm{Re}(z)|}{R^2} && \text{(integration and } e^{\lambda x}\leq 1 \text{ for } x\leq 0) \\ &\leq \frac{1}{|\mathrm{Re}(z)|}\frac{2|\mathrm{Re}(z)|}{R^2} = \frac{2}{R^2}.\end{aligned}$$

Thus by the ML inequality we get

$$|I_2(R)| \leq (1/2\pi)(2/R^2)(\pi R) = 1/R.$$

Finally, consider $|I_1(R)|$. This will be the trickiest to bound since we only know that on γ_R, G is an analytic extension of the explicitly defined G in the right half plane. First choose a constant $M(R) > 0$ such that $|G(z)| \leq M(R)$ for $z \in \gamma_R^-$. Choose δ_1 such that $0 < \delta_1 < \delta(R)$ and break up the integral defining $I_1(R)$ into two parts, corresponding to $\mathrm{Re}(z) < -\delta_1$ and $\mathrm{Re}(z) \geq -\delta_1$. This is depicted in Figure 3.

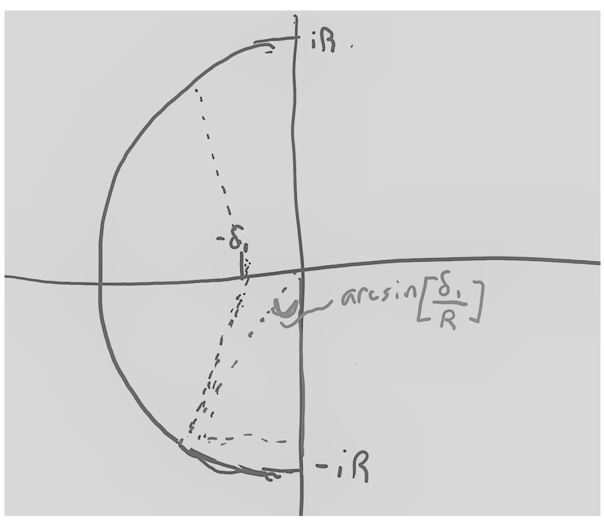

Figure 3: Splitting γ_R^-.

We know the leftmost part of the contour, where Re(z) $\le -\delta_1$, will have length less than that of the full semicircle πR, so we can bound the contribution to $|I_2(R)|$ from this part of the contour by

$$\frac{1}{2\pi}M(R)e^{-\lambda\delta_1}\ \left(\frac{1}{\delta(R)}+\frac{1}{R}\right)\pi R=\frac{1}{2}RM(R)\ \left(\frac{1}{\delta(R)}+\frac{1}{R}\right)e^{-\lambda\delta_1}$$

which for fixed R and δ_1 tends to 0 as $\lambda\to\infty$. The second contribution to $|I_2(R)|$ along the contour where $(-\delta_1 \le$ Re(z) ≤ 0 is bounded in modulus by

$$\frac{1}{2\pi}M(R)\left(\frac{1}{\delta(R)}+\frac{1}{R}\right)2R\arcsin\frac{\delta_1}{R}$$

where last factor arising from summing the lengths of the two short circular arcs on contour depicted in Figure 3. Thus for fixed R and $\delta(R)$ we can make the above expression as small as we please by taking δ_1 sufficiently close to 0.

We are finally ready to prove the conclusion of this theorem. Let $\epsilon>0$ be given. Take $R=4/\epsilon$ and fix $\delta(R), 0<\delta(R)<R$, such that G is analytic inside and on γ_R. Then as we saw above, for all λ,

$$|\frac{1}{2\pi i}\int_{\gamma_R^+}(G(z)-G_\lambda(z))e^{\lambda z}\left(\frac{1}{z}+\frac{z}{R^2}\right)dz|\le\frac{1}{R}=\frac{\epsilon}{4}$$

and

$$|\frac{1}{2\pi i}\int_{\gamma_R^-}(G_\lambda(z)e^{\lambda z}\left(\frac{1}{z}+\frac{z}{R^2}\right)dz|\le\frac{1}{R}=\frac{\epsilon}{4}.$$

Now choose δ_1 such that $0<\delta_1<\delta(R)$ and such that

$$\frac{1}{2\pi}M(R)\left(\frac{1}{\delta(R)}+\frac{1}{R}\right)2R\arcsin\frac{\delta_1}{R}<\frac{\epsilon}{4}.$$

Since we have

$$\frac{1}{2}RM(R)\left(\frac{1}{\delta(R)}+\frac{1}{R}\right)e^{-\lambda\delta_1}<\frac{\epsilon}{4}$$

for all λ sufficiently large, say $\lambda\ge\lambda_0$, it follows that

$$|G_\lambda(0)-G(0)|<\epsilon,\ \lambda\ge\lambda_0.$$ ■

4.2 The Mellin Transform

Theorem 8 Let f be a nonnegative, piecewise continuous and nondecreasing function on $[1,\infty)$ such that $f(x)=O(x)$. Then its *Mellin Transform*

$$g(z)=z\int_1^\infty f(x)x^{-z-1}dx$$

exists for $\mathrm{Re}(z) > 1$ and defines an analytic function g. Also assume that for some constant c, the function

$$g(z) - \frac{c}{z-1}$$

has an analytic extension to a neighborhood of the line $\mathrm{Re}(z) = 1$. Then as $x \to \infty$,

$$\frac{f(x)}{x} \to c.$$

Proof: Let $f(x)$ and $g(z)$ be as in the statement of the theorem above. Define F on $[0, +\infty)$ by

$$F(t) = e^{-t} f(e^t) - c.$$

Then F satisfies is bounded and piecewise continous on $[0, +\infty)$ so the first part of the hypothesis of the Tauberian Theorem 7 is satisfied. Now let us consider its Laplace transform,

$$G(z) = \int_0^\infty (e^{-t} f(e^t) - c) e^{-zt} dt.$$

Using the change of variables $x = e^t$ and $dx = e^t dt$, this becomes

$$\begin{aligned} G(z) &= \int_1^\infty \left(\frac{1}{x} f(x) - c \right) x^{-z} \frac{dx}{x} \\ &= \int_1^\infty f(x) x^{-z-2} dx - c \int_1^\infty x^{-z-1} dx \\ &= \int_1^\infty f(x) x^{-z-2} dx - \frac{c}{z} && \text{(direct integration)} \\ &= \frac{g(z+1)}{z+1} - \frac{c}{z} && \text{(by the definition of the Mellin transform shifted)} \\ &= \frac{1}{z+1} \left[g(z+1) - \frac{c}{z} - c \right]. \end{aligned}$$

Since by hypothesis we assume $g(z) - c/(z-1)$ has an analytic extension to a neighborhood of the line $\mathrm{Re}(z) = 1$ then it follows that $g(z+1) - (c/z)$ has an analytic extension to a neighborhood of the line $\mathrm{Re}(z) = 0$. Consequently the same is true of the above function G. Thus the hypotheses of the Tauberian Theorem 7 are satisfied, and we conclude that the improper integral $\int_0^\infty F(t) dt$ exists and converges to $G(0)$. Writing this in terms of f, we have $\int_0^\infty (e^{-t} f(e^t) - c) dt$ exists, or equivalently (reusing the change of variables $x = e^t$) that

$$\int_1^\infty \left(\frac{f(x)}{x} - c \right) \frac{dx}{x}$$

exists. Recalling that f is nondecreasing, we can show $f(x)/x \to c$ as $x \to \infty$. To see this, let $\epsilon > 0$ be given, and suppose that for some $x_0 > 0$, $[f(x_0)/x_0] - c \geq 2\epsilon$. It follows that

$$f(x) \geq f(x_0) \geq x_0(c+2\epsilon) \geq x(c+\epsilon) \quad \text{for } x_0 \leq x \leq \frac{c+2\epsilon}{c+\epsilon}x_0.$$

Thus we can write

$$\begin{aligned}\int_{x_0}^{\frac{c+2\epsilon}{c+\epsilon}x_0} \left(\frac{f(x)}{x} - c\right)\frac{dx}{x} &\geq \int_{x_0}^{\frac{c+2\epsilon}{c+\epsilon}x_0} \left(\frac{x(c+\epsilon)}{x} - c\right)\frac{dx}{x} \\ &\geq \int_{x_0}^{\frac{c+2\epsilon}{c+\epsilon}x_0} \frac{\epsilon}{x}dx \\ &= \epsilon \ln\left(\frac{c+2\epsilon}{c+\epsilon}\right) .\end{aligned}$$

However, we know that $\displaystyle\int_{x_1}^{x_2} \left(\frac{f(x)}{x} - c\right)\frac{dx}{x} \to 0$ as $x_1, x_2 \to \infty$, because the same integral from 1 to ∞ is convergent. Thus there exists N such that for all $x_0 > N$ we have

$$\int_{x_0}^{\frac{c+2\epsilon}{c+\epsilon}x_0} \left(\frac{f(x)}{x} - c\right)\frac{dx}{x} < \epsilon \ln\left(\frac{c+2\epsilon}{c+\epsilon}\right) .$$

But from the assumption that $[f(x_0)/x_0] - c \geq 2\epsilon$, we deduced the opposite inequality for arbitrary x_0. We must conclude that for all x_0 greater than this N, we have $[f(x_0)/x_0] - c < 2\epsilon$.

Similarly suppose $[f(x_0)/x_0] - c > -2\epsilon$ for all x_0 sufficiently large. This time we can use the inequality

$$f(x) \leq f(x_0) \leq x_0(c-2\epsilon) \leq x(c-\epsilon) \quad \text{for } \left(\frac{c-2\epsilon}{c-\epsilon}\right)x_0 \leq x \leq x_0$$

and with the same argument as before, but now with limits of integration from $\dfrac{c-2\epsilon}{c-\epsilon}x_0$ to x_0, we will also reach a contradiction. Thus we must have $|[f(x)/x] - c| < 2\epsilon$ as $x \to \infty$. Or equivalently $f(x)/x \to c$ as $x \to \infty$. ■

4.3 The Proof

We will restate our definition of the Chebyshev function ψ from Eq. 6 once more,

$$\psi(x) = \sum_{p \leq x} \left[\frac{\ln x}{\ln p}\right] \ln p.$$

Notice that ψ is a nonnegative, piecewise continuous, nondecreasing function on $[1, \infty)$. Furthermore, by Theorem 5 we have that $\psi(x) = O(x)$, thus we may take $f = \psi$ in Theorem 8 and consider the Mellin transform

$$g(z) = z\int_1^\infty \psi(x)x^{-z-1}dx.$$

By Theorem 6 we know this Mellin transform to be precisely the negative logarithmic derivative of ζ, that is $g(z) = -\zeta'(z)/\zeta(z)$. Also we have concluded that $\frac{\zeta'(z)}{\zeta(z)} + \frac{1}{z-1}$ has an analytic extension to a neighborhood of each point of $\mathrm{Re}(z) = 1$, and thus

$$g(z) - \frac{1}{z-1} = -\left[\frac{\zeta'(z)}{\zeta(z)} + \frac{1}{z-1}\right]$$

also has this analytic extension. Therefore we can conclude by Theorem 8 that $\psi(x)/x \to 1$ as $x \to \infty$, which by Theorem 4 is equivalent to the Prime Number Theorem. ■

5 The History of the Prime Number Theorem

Conjectures of the asymptotic behavior of $\pi(x)$ dates back to Gauss and Legendre in the late 18th century. In fact Gauss believed the asymptotically equivalent but more accurate

$$\pi(x) \sim \mathrm{Li(x)} := \int_2^\infty \frac{\mathrm{dx}}{\log \mathrm{x}}.$$

This is only more accurate in the sense that $|\pi(x) - \mathrm{Li(x)}|$ grows more slowly than $|\pi(x) - \frac{x}{\log x}|$ as $x \to \infty$. The graphs in Figure 4 compare the behavior of these functions. It was not until a century after Gauss and Legendre's conjec-

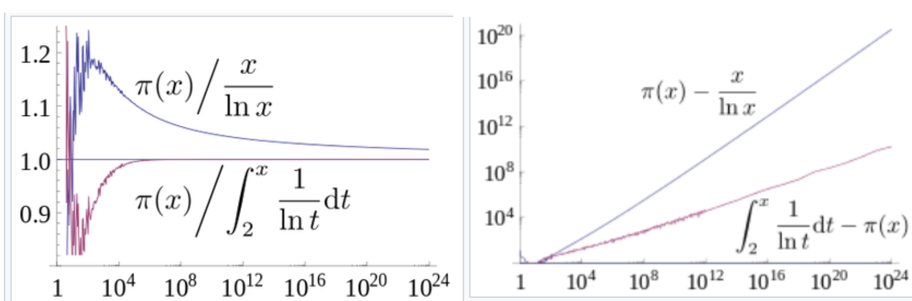

Figure 4: Asymptotic error of two approximations of $\pi(x)$ [4]

.

tures that the Prime Number Theorem was proved independently by Hadamard and Vallee Poussin in 1896 [5]. Their proofs were based on the recent work of

Riemann who showed a connection between the the zeros of the Zeta function and the distribution of primes as well as methods from complex analysis. In 1949 Atle Selberg and Paul Erdos developed proofs of the Prime Number Theorem not relying on complex analysis [6]. The version of the proof given here was first done much more recently in 1980 by D.J. Newman [7].

Newman's proof is the shortest and arguably simplest proof of the Prime Number Theorem to date (some consider it more complex because of the reliance on Cauchy's Integral Formula) [6]. A notable feature of Newman's proof that helps make it much more brief than previous proofs is the use of the "Tauberian" Theorem presented in 4.1. This theorem and similar results are named for Alfred Tauber, who is credited with proving the first theorem of this kind in the 19th century [3].

Before we conclude this paper, we will now briefly explore one of the most enthralling and yet still unresolved mathematical conjectures that involves the Riemann zeta function we used throughout this proof.

6 The Riemann Hypothesis

The Riemann hypothesis is the conjecture that the Riemann zeta function has its zeros only at the negative even integers as well as along the line $\mathrm{Re(z)} = \frac{1}{2}$. The zeros at the negative even integers are sometimes called the trivial zeros. In our proof so far we have only found an analytic extension to $\mathrm{Re(z)} > 0$. To say anything about the values for complex number with negative real values, we must first find an analytic extension to the rest of the complex plane. This can be done by showing that ζ satisfies the functional equation

$$\zeta(z) = 2^z \pi^{z-1} \sin(\frac{\pi z}{2}) \Gamma(1-z) \zeta(1-z).$$

Then one may define $\zeta(z)$ for the remaining complex numbers $\mathrm{Re(z)} \leq 0$ and $z \neq 0$ by applying this functional equation. The zeros along the negative even integers are *now* trivial to see as $\sin(\pi z/2) = 0$ whenever z is a negative even integer (The observant reader might ask why this argument does not hold for the positive even integers, however at these values the simple zeros of $\sin(\frac{\pi z}{2})$ are cancelled by the simple poles of the gamma function at $\Gamma(1-z)$).

This functional equation also implies that ζ has no other zeros with negative real part other than these trivial zeros. We have also shown in Theorem 3 that ζ has no zeros for $\mathrm{Re(z)} \geq 1$. Thus all that is left to prove the Riemann conjecture is to show no other zeros exist in the so called critical strip $0 < \mathrm{Re(z)} < 1$ except $\mathrm{Re(z)} = 1/2$. How hard can that be?

Acknowledgements

Thanks to Professor Vitaliy Lorman for encouraging me to pursue such an excellent application of complex analysis, and for his help in editing this paper. Thanks to the faculty in the Illinois Math department for their detailed proof of the Prime Number Theorem that was the basis of this work [3].

References

[1] Paul Hoffman. *The man who only loved numbers: the story of Paul Erdös and the search for mathematical truth.* [sn], 1998.

[2] Jørgen Veisdal. "The Riemann Hypothesis, explained". In: *Medium* (Apr. 2020).

[3] Illinois University Math Faculty. "Complex Analysis". In: 3–7 (Sept. 2006).

[4] Docetzee. *Prime Number Theorem Absolute Error.* Mar. 2013.

[5] Larry J Goldstein. "A history of the prime number theorem". In: *The American Mathematical Monthly* 80.6 (1973), pp. 599–615.

[6] Dorian Goldfeld. "The elementary proof of the prime number theorem: An historical perspective". In: (2004), pp. 179–192.

[7] Don Zagier. "Newman's short proof of the prime number theorem". In: *The American Mathematical Monthly* 104.8 (1997), pp. 705–708.

Appendix

This appendix includes the statement and proof of a few theorems used throughout the paper.

Theorem 9 Suppose $[a,\ b] \subseteq \mathbb{R}$, and let φ be a continuous complex-valued function on the product space $\Omega \times [a,\ b]$. Assume that for each $t \in [a, b]$, the function $z \to \varphi(z,\ t)$ is analytic on Ω. Define F on Ω by $F(z) = \int_a^b \varphi(z,\ t)dt, z \in \Omega$. Then F is analytic on Ω and

$$F'(z) = \int_a^b \frac{\partial \varphi}{\partial z}(z,\ t)dt,\ z \in \Omega.$$

Proof: Fix any disk $D(z_0,\ r)$ such that $\overline{D}(z_0,\ r) \subseteq \Omega$. Then for each $z \in D(z_0,\ r)$

we have

$$
\begin{aligned}
F(z) &= \int_a^b \varphi(z,\ t)dt \\
&= \frac{1}{2\pi i}\int_a^b \left(\int_{C(z_0,\gamma)} \frac{\varphi(w,t)}{w-z}dw\right) dt \qquad \text{(by Cauchy's Integral Formula)} \\
&= \frac{1}{2\pi i}\int_{C(z_0,r)} \left(\int_a^b \varphi(w,\ t)dt\right) \frac{1}{w-z}dw.
\end{aligned}
$$

Where the last equality can be justified by writing the path integral as an ordinary definite integral and observing that the interchange in the order of integration follows from being allowed to interchange integrals over rectangles for continuous functions.

Now $\displaystyle\int_a^b \varphi(w,\ t)dt$ is a continuous function of w since we assumed the continuity of φ on $\Omega \times [a,\ b]$), hence by the generalized Cauchy Integral Formula, F is analytic on $D(z_0,\ r)$ and for each $z \in D(z_0,\ r)$,

$$
\begin{aligned}
F'(z) &= \frac{1}{2\pi i}\int_{C(z_0,r)} \left(\int_a^b \varphi(w,\ t)dt\right) \frac{1}{(w-z)^2}dw \\
&= \int_a^b \left(\frac{1}{2\pi i}\int_{C(z_0,r)} \frac{\varphi(w,t)}{(w-z)^2}dw\right) dt \\
&= \int_a^b \frac{\partial\varphi}{\partial z}(z,\ t)dt.
\end{aligned}
$$

Where the last equality also follows from the generalized Cauchy Integral Formula. ■

Theorem 10 ***"The modulus of the integral is less than the integral of the modulus"***

Let $[a, b]$ be a closed real interval, and $f : [a, b] \to \mathbb{C}$ be a continuous complex function. Then

$$\left|\int_a^b f(t)dt\right| \leq \int_a^b |f(t)|dt$$

Proof: Define $z \in \mathbb{C}$ as the value of $\int_a^b f(t)dt$. Let $r \in [0, \infty)$ be the modulus of z (which is precisely the modulus we want to bound), and $\theta \in [0, 2\pi)$ be the argument of z. Now we can write z as $re^{i\theta}$, and then

$$\begin{aligned} r &= ze^{-i\theta} \\ &= \int_a^b e^{-i\theta} f(t) \\ &= \int_a^b \operatorname{Re}\left(e^{-i\theta} f(t)\right) + i \int_a^b \operatorname{Im}\left(e^{-i\theta} f(t)\right) \end{aligned}$$

Where in the last line we have split the integral into complex and imaginary parts in the standard way. Since r is just a real number we have

$$\mathrm{Im}\,(\mathrm{r}) = \int_\mathrm{a}^\mathrm{b} \mathrm{Im}\left(\mathrm{e}^{-i\theta}\mathrm{f}\,(\mathrm{t})\right) \mathrm{dt} = 0$$

Thus,

$$\begin{aligned} r &= \int_a^b \operatorname{Re}\left(e^{-i\theta} f(t)\right) \\ &= \int_a^b \left|\operatorname{Re}\left(e^{-i\theta} f(t)\right)\right| && \text{(modulus of real number)} \\ &= \int_a^b \left|e^{-i\theta} f(t)\right| \\ &= \int_a^b |f(t)| \end{aligned}$$

And since again r was precisely the integral we were trying to bound, this concludes the proof. ■

Theorem 11 (**Vitali's Theorem**) Let $\{f_n\}$ be a bounded sequence in $A(\Omega)$ where Ω is connected. Suppose that $\{f_n\}$ converges pointwise on $S \subseteq \Omega$ and S has a limit point in Ω. Then $\{f_n\}$ is uniformly Cauchy on compact subsets of Ω, hence uniformly convergent on compact subsets of Ω to some $f \in A(\Omega)$.

Proof: See Chapter 5 of Illinois Complex Analysis.

Theorem 12 (**ML Inequality**) If $f(z)$ is a complex-valued, continuous function on the contour Γ and if its modulus is bounded by a constant M for all z on Γ, then

$$\left|\int_\Gamma f(z)\,dz\right| \leq M\,l(\Gamma)$$

where $l(\Gamma)$ is the arc length of Γ.

Proof: Let $\gamma(t)$ be a parameterization of the contour with $\alpha \leq\ t \leq \beta$, then the proof follows simply from

$$\begin{aligned}\left|\int_\Gamma f(z)\,dz\right| &= \left|\int_\alpha^\beta f(\gamma(t))\gamma'(t)\,dt\right| \\ &\leq \int_\alpha^\beta |f(\gamma(t))|\,|\gamma'(t)|\;dt \\ &\leq M\int_\alpha^\beta |\gamma'(t)|\;dt = M\,l(\Gamma).\end{aligned}$$

■